

Non-recurrent parameter rays of the Mandelbrot set*

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Abstract

In this paper, we prove that any parameter ray at a non-recurrent angle θ lands at a non-recurrent parameter c with θ a characteristic angle of f_c ; and conversely, every non-recurrent parameter c is the landing point of one or two parameter rays at non-recurrent angles, and these angles are exactly the characteristic angles of f_c .

1 Introduction

The quadratic family $\{f_c : z \mapsto z^2 + c\}$ exhibits rich dynamics, when iterated. The Mandelbrot set

$$\mathcal{M} := \{c \in \mathbb{C} \mid f_c^n(c) \not\rightarrow \infty \text{ as } n \rightarrow \infty\}$$

organizes the space of quadratic polynomials up to conjugacy and has a beautiful structure. It has been a very active area of research in the past few decades. The importance of the Mandelbrot set is due to the fact that it is the simplest non-trivial parameter space of analytic families of iterated holomorphic maps, and because of its universality as explained in [DH1, Mc]

Much of the topological and combinatorial structures of the Mandelbrot set has been discovered by the work of Douady and Hubbard [DH2]. A fundamental result in [DH2] is to describe the landing behavior of the rational parameter rays. One can see also [Mil2, Sch, PR] for alternative approaches.

In this article, we study the landing property of irrational, precisely the *non-recurrent*, parameter rays.

Set $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and $\tau : \mathbb{T} \rightarrow \mathbb{T}$ the angle doubling map, i.e., $\tau(\theta) = 2\theta \bmod \mathbb{Z}$, $\theta \in \mathbb{T}$. By abuse of notations, we identify \mathbb{T} with the unit circle under the correspondence $t \mapsto e^{2\pi it}$.

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An angle $\theta \in \mathbb{T}$ is called *non-recurrent* if $\tau^n(\theta) \neq \tau^m(\theta)$ for any $0 \leq m < n$, and there exists $\delta > 0$ such that $|\theta - \tau^n(\theta)| > \delta$ for all $n \geq 1$. A quadratic polynomial f_c , or the parameter c , is called *non-recurrent* if $c \in \mathcal{M}$, all periodic points of f_c are repelling and $|f_c^n(0)| > \delta$ for all $n \geq 1$ and a positive constance δ .

It is known that the Julia sets of non-recurrent quadratic polynomials are connected and locally-connected. The angle of an external ray landing at the critical value c is said to be a *characteristic angle* of f_c . Refer to Section 2 for the definitions of external rays and parameter rays. The following is our main theorem.

Theorem 1.1. *Any parameter ray at a non-recurrent angle lands at a non-recurrent parameter c such that θ is a characteristic angle of f_c . Conversely, every non-recurrent parameter c is the landing point of one or two parameter rays at non-recurrent angles, and these angles are exactly the characteristic angles of f_c .*

Our proof is based on Kiwi's Combinatorial Continuity Theorem [Ki2, Theorem 1] and Yoccoz Rigidity Theorem [Hu, Theorem III] (or [Ze, Theorem 4.1]). We will review some background of polynomial dynamics and fix notations in Section 2, and introduce the concept of *real lamination* in Section 3. In Section 4, we verify two combinatorial results used in the proof of the main theorem, and the proof of Theorem 1.1 is left in Section 5.

2 Polynomial dynamics

One can refer to [DH2] for the details of the content in this section.

Let $f_c(z) = z^2 + c$, $z \in \mathbb{C}$ be a quadratic polynomial. The set of all points which remain bounded under all iterations of f_c is called the *Filled-in Julia set* K_c . The boundary of the Filled-in Julia set is defined to be the *Julia set* J_c and the complement of the Julia set is defined to be its *Fatou set* F_c .

If $c \in \mathcal{M}$, the filled-in Julia set K_c is *simply-connected*, i.e., $\mathbb{C} \setminus K_c$ is connected. There is then a unique biholomorphic map ϕ_c from $\mathbb{C} \setminus K_c$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$, called the *Böttcher coordinate*, such that $\lim_{z \rightarrow \infty} \phi_c(z)/z = 1$ and $\phi_c \circ f_c(z) = \phi_c(z)^2$ for $z \in \mathbb{C} \setminus K_c$. The preimage of $(1, \infty)e^{2\pi i t}$ under ϕ_c , denoted $\overline{R_c(t)}$, is called the *external ray at angle t* . We say that the external ray $R_c(t)$ *lands* if $\overline{R_c(t)} \cap K_c$ is a singleton, and this point, denoted by $\gamma_c(t)$, is called the *landing point* of $R_c(t)$.

In the parameter plane, the Mandelbrot set \mathcal{M} is simply-connected, and there is a biholomorphic map Φ from $\mathbb{C} \setminus \mathcal{M}$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$. The *parameter ray at angle θ* is defined as the set $R_{\mathcal{M}}(\theta) := \{\Phi^{-1}(re^{2\pi i \theta}), r > 1\}$. Similarly, if $\overline{R_{\mathcal{M}}(\theta)} \cap \mathcal{M}$ is a singleton, we say that $R_{\mathcal{M}}(\theta)$ *lands*.

3 The impression and real lamination of polynomials

Let f_c be a quadratic polynomial with connected Julia set. If J_c is locally connected, the map ϕ_c^{-1} can be continuously extended to $\mathbb{C} \setminus \mathbb{D}$ and each external ray $R_c(t)$ lands at a Julia point. The map $\gamma_c : \mathbb{T} \rightarrow J_f$, $t \mapsto \gamma_c(t)$, is continuous and surjective. In this case, the landing pattern of external rays for f_c induces an equivalence relation $\lambda(c)$ on \mathbb{T} such that $t \stackrel{\lambda(c)}{\sim} s$ if and only if $\gamma_c(t) = \gamma_c(s)$.

Kiwi [Ki2] generalized the definition of such an equivalence relation to a class of non locally-connected case, with the concept *impression* instead of the landing point in the locally-connected case.

We still assume that f_c has the connected Julia set. Consider an argument $t \in \mathbb{T}$. We say that $z \in J_f$ belongs to the *impression* of t , written $\text{Imp}_c(t)$, if and only if there exists a sequence $\{z_n \in R_c(t_n)\}$ converging to z , with $\{t_n\} \subset \mathbb{T}$ converging to t . Note that $f_c(\text{Imp}_c(t)) = \text{Imp}_c(\tau(t))$, and $\text{Imp}_c(t) = \gamma_c(t)$ for each $t \in \mathbb{T}$ if J_c is locally connected.

Similar to the locally-connected case, we have the following two facts about the impressions, which will be used in the proof of Proposition 5.1

Lemma 3.1. 1. *If there exists a sequence $\{z_n \in \text{Imp}_c(t_n)\}$ with $z_n \rightarrow z$ and $t_n \rightarrow t$ as $n \rightarrow \infty$, then $z \in \text{Imp}_c(t)$.*

2. *For any $z \in J_c$, there exists an argument $t \in \mathbb{T}$ with $z \in \text{Imp}_c(t)$.*

Proof. 1. For each n , since $z_n \in \text{Imp}_c(t_n)$, we can choose an argument s_n and a point $w_n \in R_c(s_n)$ such that $|z_n - w_n| < |z_n - z|$ and $|t_n - s_n| < |t_n - t|$. Then we have $|w_n - z| < 2|z_n - z| \rightarrow 0$ and $|s_n - t| < 2|t_n - t| \rightarrow 0$ as $n \rightarrow \infty$. It means that $w_n \in R_c(s_n)$ converges to z and s_n converges to t , hence $z \in \text{Imp}_c(t)$.

2. Let $z \in J_c$. Since J_c is the boundary of the basin $\mathbb{C} \setminus K_c$, there exists a sequence $\{z_n\} \subset \mathbb{C} \setminus K_c$ with z_n converging to z . Each z_n belongs to an external ray of argument t_n . By picking a subsequence, we assume $t_n \rightarrow t$ as $n \rightarrow \infty$. It follows from the definition that $z \in \text{Imp}_c(t)$. \square

Let f_c be a quadratic polynomial with connected Julia set and without irrational neutral cycles. Following Kiwi (see [Ki2, Definition 2.2]), the *real lamination* of f_c is the smallest equivalence relation $\lambda(c)$ in \mathbb{T} which identifies s and t whenever $\text{Imp}_c(s) \cap \text{Imp}_c(t) \neq \emptyset$. For a $\lambda(c)$ -class A , we denote $\text{Imp}_c(A)$ the union of the impressions $\text{Imp}_c(t)$ for all $t \in A$.

4 The equivalence relation generated by angles

For any angle $\theta \in \mathbb{T}$, its preimages $\{\theta/2, (\theta+1)/2\}$ under τ divide \mathbb{T} into two closed half circles, which are denoted by L_0, L_1 respectively. Then we can endow each angle $t \in \mathbb{T}$

two itineraries $\iota_\theta^\pm(t)$ with respect to θ such that $\iota_\theta^\pm(t) = i_0 i_1 \dots$ if, for each $n \geq 0$, there exists $\epsilon > 0$ with $(\tau^n(t), \tau^n(t) \pm \epsilon) \subset L_{i_n}$.

From the definition, we can see that if t is not an iterated preimage of θ , then $\iota_\theta^+(t) = \iota_\theta^-(t)$. In particular, we have $\iota_\theta^+(\theta) = \iota_\theta^-(\theta)$ for any non-periodic θ . In this case, the sequence $\iota_\theta^+(\theta) = \iota_\theta^-(\theta)$ is called the *kneading sequence of θ* , written $\nu(\theta)$. It is called *aperiodic* if it is not a periodic symbol sequence under the shift map.

By Kiwi [Ki2, Definition 4.5], the *equivalence relation generated by $\theta \in \mathbb{T}$* , denoted by $\lambda(\theta)$, is defined as the smallest equivalence relation such that if $\iota_\theta^+(t) = \iota_\theta^-(s)$, then s and t are equivalent.

From now on, we always assume that $\theta \in \mathbb{T}$ is non-recurrent. The following is a key Lemma in our proof.

Lemma 4.1. *If θ is non-recurrent, then $\nu(\theta)$ is aperiodic.*

Proof. On the contrary, we assume that $\nu(\theta)$ is periodic of period $p \geq 1$. For each integer $0 \leq k \leq p$, set

$$B_k := \{\tau^{k+np}(\theta) \mid n \geq 0\}.$$

Then we have $\tau(\overline{B_k}) \subset \overline{B_{k+1}}$, $k \in \{0, \dots, p-1\}$, and $\overline{B_p} \subset \overline{B_0}$. Note that all elements of B_k have a common itinerary, then each B_k is contained in a component of $\mathbb{T} \setminus \{\theta/2, (\theta+1)/2\}$. Moreover, by the non-recurrent property, the closures $\overline{B_k}$ are disjoint from $\{\theta/2, (\theta+1)/2\}$. It follows that for each $k \in \{0, \dots, p-1\}$, the map $\tau : \overline{B_k} \rightarrow \overline{B_{k+1}}$ is injective, and hence $\tau^p : \overline{B_0} \rightarrow \overline{B_0}$ is injective. According to [Mil1, Lemma 18.8], the set $\overline{B_0}$ is finite, a contradiction. \square

Since $\nu(\theta)$ is aperiodic, by [Ki2, Proposition 4.7], we get that

Proposition 4.2. *The equivalence relation $\lambda(\theta)$ is closed and satisfies that*

1. *each $\lambda(\theta)$ -class is a finite subset of \mathbb{T} ;*
2. *if A is a $\lambda(\theta)$ -class, then $\tau(A)$ is a $\lambda(\theta)$ -class;*
3. *for any two different $\lambda(\theta)$ -classes A, B , the convex hulls of A and B are disjoint.*

Combining the fact that θ is non-recurrent, we can obtain more information about $\lambda(\theta)$. Since θ is not periodic, then $\iota_\theta^+(\theta/2) = \iota_\theta^-(\theta/2)$, and hence $\theta/2$ and $(\theta+1)/2$ are $\lambda(\theta)$ -equivalent. We call the $\lambda(\theta)$ -class containing $\{\theta/2, (\theta+1)/2\}$ the *critical class*, denoted by C_θ ; and the one containing θ the *characteristic class*, denoted by A_θ .

Lemma 4.3. *Let A_θ be the characteristic class of $\lambda(\theta)$. Then we have*

1. *A_θ is wandering, i.e., $\tau^n(A_\theta) \cap \tau^m(A_\theta) = \emptyset$ for each $0 \leq m < n$;*
2. *A_θ contains at most two angles;*

3. A_θ is non-recurrent, i.e., $\exists \delta > 0$ s.t $\text{dist}(A_\theta, \tau^n(A_\theta)) > \delta$ for all $n \geq 1$.

Proof. 1. On the contrary, without loss of generality, we assume that A_θ is periodic. Then the fact of $\#A_\theta < \infty$ implies that θ is eventually periodic, a contradiction.

2. Since A_θ is wandering, its orbit does not contain C_θ . Note that each $\lambda(\theta)$ -class except C_θ is contained in one component of $\mathbb{T} \setminus \{\theta/2, (\theta+1)/2\}$ (by 3 of Proposition 4.2), it follows that $\#A_\theta = \#\tau^n(A_\theta)$ for all $n \geq 0$. Using Thurston's No Wandering Polygon Theorem ([Thu, Theorem II.5.2]), the conclusion holds.

3. If A_θ contains one angle, since θ is non-recurrent, the set A_θ is naturally non-recurrent. So, by assentation (2), we just need to prove the case that $\#A_\theta = 2$.

For any $\alpha \neq \beta \in \mathbb{T}$, we define the arc $(\alpha, \beta) \subset \mathbb{T}$ as the closure of the connected component of $\mathbb{T} \setminus \{\alpha, \beta\}$ that consists of the angles we traverse if we move on \mathbb{T} in the counterclockwise direction from α to β . The length of an arc $S \subset \mathbb{T}$ is denoted by $|S|$. We define a map σ on all arcs in \mathbb{T} such that $\sigma(\alpha, \beta)$ equals to $(\tau(\alpha), \tau(\beta))$ if $\tau(\alpha) \neq \tau(\beta)$, and equals to $\tau(\alpha)$ otherwise. It is apparent that $|\sigma(S)| = 2|S|$ if $|S| < 1/2$ and $|\sigma(S)| = 2|S| - 1$ otherwise.

Let $A_\theta = \{\theta, \eta\}$. Then it divides \mathbb{T} into two closed arcs. We denote the shorter one by S_1^+ and the longer one by S_1^- . For $n \geq 1$, set

$$A_n := \tau^{n-1}(A_\theta), \quad S_n^+ := \sigma^{n-1}(S_1^+) \text{ and } S_n^- := \sigma^{n-1}(S_1^-).$$

According to the proof of assentation 2, each A_n is a $\lambda(\theta)$ -class containing two angles. And it divides \mathbb{T} into S_n^+ and S_n^- . By 2 of Proposition 4.2, the critical class C_θ is equal to $\{\frac{\theta}{2}, \frac{\theta+1}{2}, \frac{\eta}{2}, \frac{\eta+1}{2}\}$. It divides \mathbb{T} into four arcs. We denote the two shorter ones by $S_0^+, -S_0^+$, and the two longer ones by $S_0^-, -S_0^-$. It is clear that $|S_0^+| = |-S_0^+| < |S_0^-| = |-S_0^-| < 1/2$ and $\sigma(\pm S_0^\delta) = S_1^\delta$ for $\delta \in \{+, -\}$.

We claim that in the set $\{S_n^\pm \mid n \geq 1\}$, the arc S_1^+ has the shortest length. If not, suppose that $k \geq 2$ is the first integer such that S_k^+ or S_k^- , say S_k^+ , has a shorter length than S_1^+ . Then $|S_{k-1}^+| > 1/2$. It follows from 3 of Proposition 4.2 that the arc S_{k-1}^+ contains C_θ , and hence contains three of the arcs $\pm S_0^+, \pm S_0^-$. Its image $\sigma(S_{k-1}^+) = S_k^+$ therefore contains either $\tau(S_0^+) = S_1^+$ or $\tau(S_0^-) = S_1^-$. It implies $|S_k^+| \geq \min\{|S_1^+|, |S_1^-|\} = |S_1^+|$, a contradiction to the assumption that $|S_k^+| < |S_1^+|$.

Set $\theta_n := \tau^n(\theta)$ and $\eta_n := \tau^n(\eta)$ for all $n \geq 1$. We now start to prove point 3 by contradiction. We can assume that $\text{dist}(A_{n_k}, A_\theta) \rightarrow 0$, $\theta_{n_k} \rightarrow \theta'$, and $\eta_{n_k} \rightarrow \eta'$ as $k \rightarrow \infty$ by passing to a subsequence if necessary. Since $\lambda(\theta)$ is closed, η', θ' are in the same $\lambda(\theta)$ -class. Notice that $|S_{n_k}^\pm| \geq |S_1^+|$ as explained in the claim above. Then $\eta' \neq \theta'$. Since $\text{dist}(A_{n_k}, A_\theta) \rightarrow 0$ and $\#A_\theta = 2$, we have $\theta' = \eta$ and $\eta' = \theta$.

Hence for any $\epsilon > 0$, there exist k_1 and k_2 such that $|\eta_{n_{k_1}} - \theta| < \epsilon/2$ and

$$2^{n_{k_1}} |\theta_{n_{k_2}} - \eta| = |\tau^{n_{k_1}}(\theta_{n_{k_2}}) - \tau^{n_{k_1}}(\eta)| < \epsilon/2.$$

It follows that $|\tau^{n_{k_1}+n_{k_2}}(\theta) - \theta| < \epsilon$ for any $\epsilon > 0$, which is impossible. \square

5 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following two propositions.

Proposition 5.1. *Given a non-recurrent angle θ , the parameter ray $R_{\mathcal{M}}(\theta)$ lands at a non-recurrent parameter c such that f_c takes θ as a characteristic angle.*

Proof. We set $\text{Acc}_{\mathcal{M}}(\theta)$ the accumulation set of $R_{\mathcal{M}}(\theta)$ on \mathcal{M} . By Lemma 4.1 and [Ki2, Theorem 1], we have that all cycles of f_c are repelling and $\lambda(c) = \lambda(\theta)$ for all $c \in \text{Acc}_{\mathcal{M}}(\theta)$. Let $c \in \text{Acc}_{\mathcal{M}}(\theta)$. We will show that f_c is non-recurrent with a characteristic angle θ .

By Lemma 3.1 (2), we choose an angle $t_0 \in \mathbb{T}$ with $0 \in \text{Imp}_c(t_0)$. Note that the Böettcher coordinate ϕ_c satisfies that $\phi_c(-z) = -\phi_c(z)$ for $z \in \mathbb{C} \setminus K_c$, then the sets $\text{Imp}_c(t_0)$ and $\text{Imp}_c(t_0 + 1/2)$ are symmetric about the origin. It follows that $0 \in \text{Imp}_c(t_0) \cap \text{Imp}_c(t_0 + 1/2)$, and hence $t_0, t_0 + 1/2$ are in a common $\lambda(\theta)$ -class (because $\lambda(c) = \lambda(\theta)$). By Proposition 4.2 (3), we know that $t_0, t_0 + 1/2$ are contained in the critical class C_θ . Then the impression of the characteristic class $\text{Imp}_c(A_\theta)$ contains the critical value c .

Set $c_n := f_c^{n-1}(c)$ and recall that $A_n := \tau^{n-1}(A_\theta)$. Then c_n belongs to $\text{Imp}_c(A_n)$ for each $n \geq 1$. The non-recurrent property of f_c is equivalent to that the accumulation set of $\{c_n, n \geq 1\}$ is disjoint from the critical value c . We continue the argument by contradiction and assume that the sequence $\{c_{n_k} \in \text{Imp}_c(t_{n_k})\}$ with $t_{n_k} \in A_{n_k}$ converges to c as $n \rightarrow \infty$.

By passing to a subsequence if necessary, we assume that $t_{n_k} \rightarrow t$ as $k \rightarrow \infty$. From Lemma 3.1(1), we see that $c \in \text{Imp}_c(t)$, and hence $t \in A_\theta$. This contradicts Lemma 4.3 (3). Thus f_c is non-recurrent. Since the Julia sets of non-recurrent quadratic polynomials are locally connected, the set $\text{Imp}_c(A_\theta)$ reduces to one point c . So $R_c(\theta)$ lands at c .

We have seen that all f_c with $c \in \text{Acc}_{\mathcal{M}}(\theta)$ are non-recurrent and have a common real lamination $\lambda(\theta)$. Due to Yoccoz Rigidity Theorem [Hu, Theorem III] or [Ze, Theorem 4.1], we have that $\text{Acc}_{\mathcal{M}}(\theta)$ reduces to one point. So the parameter ray $R_{\mathcal{M}}(\theta)$ lands. \square

Proposition 5.2. *Let f_c be a non-recurrent quadratic polynomial. Then it has at most two characteristic angles, and the parameter rays at these angles land at c .*

Proof. Let f_c be a non-recurrent quadratic polynomial. Since J_c is locally connected, there exists a characteristic angle θ with $R_c(\theta)$ landing at c . Clearly θ is non-recurrent. By Proposition 5.1, the parameter ray $R_{\mathcal{M}}(\theta)$ lands at a non-recurrent parameter c' so that $f_{c'}$ has a characteristic angle θ . We then have that $\theta/2$ and $(\theta+1)/2$ are contained in both a $\lambda(c)$ -class and a $\lambda(c')$ -class. By [Ki2, Proposition 4.10], we get $\lambda(c) = \lambda(c') = \lambda(\theta)$. Using again the Yoccoz Rigidity Theorem, it follows that $c = c'$. By Lemma 4.3 (2), the cardinality of the $\lambda(c)$ -class that contains θ is at most two. Hence f_c has at most two characteristic angles. \square

Proof of Theorem 1.1. It follows directly from Propositions 5.1 and 5.2. \square

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